From a car-following model with reaction time to a macroscopic convection-diffusion traffic flow model

September 28, 2016 | Antoine Tordeux[2] | Forschungszentrum Jülich, Germany

[2] a.tordeux@fz-juelich.de

Outline

Microscopic model

Micro–Macro derivation

Numerical schemes

Stability analysis

Simulation results

Outline

Pursuit law

The microscopic model comes from the generalized Newell (1961) one

$$\dot{x}_i(t+\tau) = W(\Delta x_i(t)), \qquad (i,t) \in \mathbb{Z} \times (0,+\infty)$$

with τ the reaction time (if positive), $\Delta x_i(t) = x_{i+1}(t) - x_i(t)$ the spacing and $W(\cdot)$ the equilibrium (or optimal) speed function

Writing τ in rhs and applying a linear approximation for small τ we get

$$\dot{x}_i(t) = W\big(\Delta x_i(t) - \tau[\dot{x}_{i+1}(t) - \dot{x}_i(t)]\big)$$

The model is then obtained by substituting the speeds $\dot{x}$ in rhs by the optimal speed $W(\Delta x)$:

$$\dot{x}_i(t) = W\Big(\Delta x_i(t) - \tau\big[W(\Delta x_{i+1}(t)) - W(\Delta x_i(t))\big]\Big) \tag{1}$$

Pursuit law

Microscopic model:

$$\dot{x}_i(t) = W\Big(\Delta x_i(t) - \tau\big[W(\Delta x_{i+1}(t)) - W(\Delta x_i(t))\big]\Big), \qquad (i,t) \in \mathbb{Z}\times(0,+\infty)$$

- Speed model with two predecessors in interaction
- Collision-free (by construction): $\Delta x_i \geq \ell \quad \forall i,t$
  if $W(\cdot) \geq 0$ and $W(s) = 0$ for all $s < \ell$
- Same linear stability condition for homogeneous solutions as original

Newell model (or OVM by Bando (1998)): $|\tau| W' < 1/2$

(Homogenization for small τ, stop-and-go for high τ)

Outline

Rewriting the microscopic model

Same methodology as in [Aw *et al.* (2002)] considering the density at vehicle i and time t, $\rho_i(t)$, as the inverse of the spacing

$$\rho_i(t) := \frac{1}{\Delta x_i(t)}. \tag{2}$$

The microscopic model becomes

$$\dot{x}_i = W\Big(\frac{1}{\rho_i(t)} - \tau\big[W\big(\frac{1}{\rho_{i+1}(t)}\big) - W\big(\frac{1}{\rho_i(t)}\big)\big]\Big) =: \tilde{V}(\rho_{i+1}, \rho_i), \tag{3}$$

Then,

$$\partial_t \frac{1}{\rho_i(t)} = \partial_t \Delta x_i(t) = \Big(\tilde{V}(\rho_{i+2}, \rho_{i+1}) - \tilde{V}(\rho_{i+1}, \rho_i)\Big). \tag{4}$$

(semi–discretized version of hyperbolic partial differential equation in the space of vehicle indices)

Derivation

$y \in \mathbb{R}$ such that $y_i = i\Delta y$ with Δy proportional to ℓ

Piecewise constant density $\rho(t,y)$ such that $\frac{1}{\rho_i(t)} = \frac{1}{\Delta y}\int_{y_i-\frac{\Delta y}{2}}^{y_i+\frac{\Delta y}{2}} \frac{1}{\rho(t,z)} dz$.

Rescaling of time $t \to t\Delta y$ and reaction time $\tau \to \tau\Delta y$ to obtain

$$\partial_t \frac{1}{\rho_i(t)} - \frac{1}{\Delta y}\left(V\Big(\frac{\rho_{i+1}}{1-\rho_{i+1}\tau\frac{Z_{i+1}}{\Delta y}}\Big) - V\Big(\frac{\rho_i}{1-\tau\rho_i\frac{Z_i}{\Delta y}}\Big)\right) = 0, \tag{5}$$

where $Z_i := V(\rho_{i+1}) - V(\rho_i)$ and $V(x) = W(\frac{1}{x})$ for $x > 0$ (non-increasing).

(5) is an upwind discretization in the rescaled time and in the limit Δy (i.e. $i \to \infty$ and $\ell \to 0$) of the macroscopic equation

$$\partial_t \frac{1}{\rho} - \partial_y V\left(\frac{\rho}{1-\tau\rho\partial_y V(\rho)}\right) = 0. \tag{6}$$

Macroscopic model in Eulerian coordinates

Coordinate transformation $(t, y) \to (t, x)$ where $y = \int_{-\infty}^{x} \rho(t, x) dx$.

In the Eulerian coordinates (t, x), the macroscopic model Eq. (6) reads

$$\partial_t \rho + \partial_x \left(\rho V \left(\frac{\rho}{1 - \tau \partial_x V(\rho)} \right) \right) = 0. \tag{7}$$

→ Extension of the LWR model with FD $\rho \mapsto V\big(\rho/(1 - \tau \partial_x V(\rho))\big)$.

Taylor expansion in terms of τ for equation (7) yields

$$\partial_t \rho + \partial_x(\rho V(\rho)) = -\tau \partial_x \left((\rho V'(\rho))^2 \partial_x \rho \right) \tag{8}$$

Note that for constant densities ρ (or $\tau = 0$) the additional term vanishes and we recover the classical LWR.

Fundamental diagram

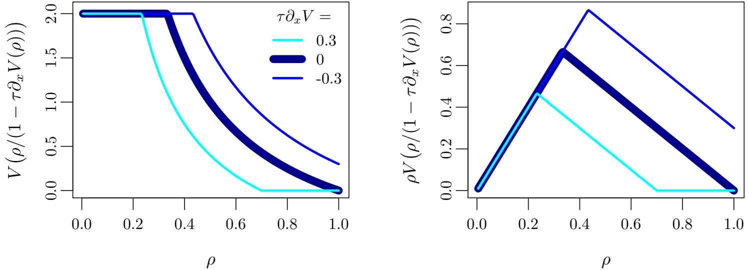


Figure: Illustration for the FD obtained in the macroscopic model with constant inhomogeneity $\tau\partial_x V(\rho) = \pm 0.3$. Triangular FD $V : \rho \mapsto \max\{\min\{2, 1/\rho - 1\}\}$. Bounded FD as in [Colombo (2003), Goatin (2006), Colombo *et al.* (2010)]

Macroscopic model

(Eulerian coordinates)

$$\underbrace{\partial_t \rho + \underbrace{\partial_x(\rho V(\rho))}_{\text{Convection}}}_{\text{LWR model}} = \underbrace{-\tau \partial_x \left((\rho V'(\rho))^2 \partial_x \rho \right)}_{\text{Diffusion}} \tag{9}$$

- $\tau < 0$: Convection-diffusion equation (LWR with diffusion – cf. Burger equations) with variable diffusion coefficient (cf. Fick equations)
- $\tau = 0$: LWR model
- $\tau > 0$: Negative diffusion??

Outline

Discrete macroscopic models

$$\rho_i(t+\delta t) = \rho_i(t) + \frac{\delta t}{\delta x}\left(f_{i-1}(t) - f_i(t)\right) \tag{10}$$

- Godunov/Euler scheme

$$f_i = G(\rho_i, \rho_{i+1}) + \frac{\tau}{\delta x}(\rho_i V'(\rho_i))^2 \left(\rho_{i+1} - \rho_i\right) \tag{D1}$$

- Simple Godunov scheme

$$f_i = G\left(\frac{\rho_i}{1 - \frac{\tau}{\delta x}\left(V(\rho_{i+1}) - V(\rho_i)\right)}, \frac{\rho_{i+1}}{1 - \frac{\tau}{\delta x}\left(V(\rho_{i+2}) - V(\rho_{i+1})\right)}\right) \tag{D2}$$

- Double Godunov scheme

$$f_i = G(\rho_i, \rho_{i+1}) + \frac{\tau}{\delta x}\rho_i V'(\rho_i)\left[G(\rho_{i+1}, \rho_{i+2}) - G(\rho_i, \rho_{i+1})\right] \tag{D3}$$

with $G(x, y) = \min\{\Delta(x), \Sigma(y)\}$ the Godunov scheme, $\Delta(\cdot)$ and $\Sigma(\cdot)$ are the demand and supply functions

Outline

Stability analysis for the continuous model

Stability analysis of the homogeneous solution where $\rho(x,t) = \rho_E$ for all x, t (ρ_E being the mean density)

Perturbation to homogeneous solution $\varepsilon(x,t) = \rho(x,t) - \rho_E$

Linearisation: $\varepsilon_t = F(\rho_E + \varepsilon, \varepsilon_x, \varepsilon_{xx}) \approx \alpha\varepsilon + \beta\varepsilon_x + \gamma\varepsilon_{xx}$

with $F(\rho, \rho_x, \rho_{xx}) = -\partial_x(\rho V(\rho)) - \tau\partial_x\big((\rho V'(\rho))^2\partial_x\big)$, $\alpha = \frac{\partial F}{\partial \rho}(\rho_E, \rho_E, \rho_E) = 0$,

$$\beta = \frac{\partial F}{\partial \rho_x} = -V(\rho_E) - \rho_E V'(\rho_E), \text{ and } \gamma = \frac{\partial F}{\partial \rho_{xx}} = -\tau(\rho_E V'(\rho_E))^2$$

Linear system: $\varepsilon = ze^{\lambda t - ixl}$, $\varepsilon_t = \lambda\varepsilon$, $\varepsilon_x = -il\varepsilon$, $\varepsilon_{xx} = -l^2\varepsilon$

$\lambda = \tau\left(l\rho_E V'(\rho_E)\right)^2 + il(V(\rho_E) + \rho_E V'(\rho_E))$ — Stable if $\Re(\lambda) < 0\ \forall l > 0$

$\rightarrow$ Homogeneous solution linearly stable if $\tau < 0$ (positive diffusion)

JÜLICH
FORSCHUNGSZENTRUM

Stability analysis for the discrete schemes

Perturbation to homogeneous solution $\varepsilon_i(t) = \rho_i(t) - \rho_E$

Linearisation of the perturbed system:

$$
\begin{aligned}
\varepsilon_i(t + \delta\mathrm{t}) &= \rho_i(t + \delta\mathrm{t}) - \rho_E = F(\rho_i(t), \rho_{i+1}(t), \rho_{i+2}(t), \rho_{i-1}(t)) - \rho_E \\
&\approx \alpha\,\varepsilon_i(t) + \beta\,\varepsilon_{i+1}(t) + \gamma\,\varepsilon_{i+2}(t) + \xi\,\varepsilon_{i-1}(t)
\end{aligned}
$$

with

$$
\begin{aligned}
\alpha &= \tfrac{\partial F}{\partial \rho_i}(\rho_E, \rho_E, \rho_E, \rho_E) & \gamma &= \tfrac{\partial F}{\partial \rho_{i+2}}(\rho_E, \rho_E, \rho_E, \rho_E) \\
\beta &= \tfrac{\partial F}{\partial \rho_{i+1}}(\rho_E, \rho_E, \rho_E, \rho_E) & \xi &= \tfrac{\partial F}{\partial \rho_{i-1}}(\rho_E, \rho_E, \rho_E, \rho_E)
\end{aligned}
$$

General conditions for stability of the discrete schemes

N cells with periodic boundary conditions — The linear dynamics are $\vec{\varepsilon}(t+\delta\mathrm{t}) = M\,\vec{\varepsilon}(t)$ with $\vec{\varepsilon} = {}^{\mathsf{T}}(\varepsilon_1,\ldots,\varepsilon_N)$ and M a sparse matrix with $(\xi,\alpha,\beta,\gamma)$ on the diagonal

If $M = PDP^{-1}$ with D a diagonal matrix, then $\vec{\varepsilon}(t) = PD^{t/\,\delta\mathrm{t}}P^{-1}\,\vec{\varepsilon}(0) \to \vec{0}$ if all the coefficients of D are less than one excepted one

M is circulant therefore the eigenvectors of M are $z(\iota^0,\iota^1,\ldots,\iota^{m-1})$ with $\iota = \exp\left(i\frac{2\pi l}{N}\right)$ and $z \in \mathbb{Z}$, and the eigenvectors are $\lambda_l = \alpha + \beta\iota_l + \gamma\iota_l^2 + \xi\iota_l^{-1}$

The system is linearly stable if $\quad |\lambda_l| < 1 \quad$ for all $l = 1,\ldots,N-1$

with $\lambda_l^2 = \alpha^2+\beta^2+\gamma^2+\xi^2-2\alpha\gamma-2\beta\xi+2f(c_l)$, $c_l = \cos(2\pi l/N)$ and $f(x) = (\alpha\beta+\alpha\xi+\beta\xi-3\gamma\xi)x + 2(\alpha\gamma+\beta\xi)x^2 + 4\gamma\xi x^3$

Stability analysis for the scheme (D1)

Affine speed function $V(\rho) = \frac{1}{T}(1/\rho - \ell)$, with $T > 0$ the time gap between the vehicles and $\ell > 0$ their size — Godunov scheme is $G(x, y) = \frac{1}{T}(1 - y\ell)$

The scheme (D1) is

$$F_1(\rho_i, \rho_{i+1}, \rho_{i+2}, \rho_{i-1}) = \rho_i + \frac{\delta t}{\delta x T}\left(\ell(\rho_{i+1} - \rho_i) + \frac{\tau}{\delta x T}\left(\frac{\rho_i - \rho_{i-1}}{\rho_{i-1}^2} - \frac{\rho_{i+1} - \rho_i}{\rho_i^2}\right)\right)$$

and (with $A = \frac{\delta t\, \ell}{\delta x\, T}$ and $B = \frac{\delta t\, \tau}{(\delta x\, T \rho_E)^2}$)

$$\begin{aligned} \alpha &= 1 - A + 2B & \qquad \beta &= A - B \\ \gamma &= 0 & \qquad \xi &= -B \end{aligned}$$

Signs of the partial derivative $\alpha\ \beta\ \gamma$

$A = \frac{\delta t\, \ell}{\delta x\, T}$ and $B = \frac{\delta t\, \tau}{(T\, \delta x\, \rho_E)^2}$

- $\alpha = 1 - A + 2B$ is positive if

$$\delta t < \frac{\delta x\, T}{\ell} \left(1 - \frac{2\tau}{T \ell\, \delta x\, \rho_E^2}\right)^{-1} \tag{P_α}$$

  if $\tau < \frac{1}{2} T \ell\, \delta x\, \rho_E^2$, or for all $\delta t \geq 0$ if $\tau \geq \frac{1}{2} T \ell\, \delta x\, \rho_E^2$
  Moreover $1 - \alpha \geq 0$ iff $\tau < \frac{1}{2} T \ell\, \delta x\, \rho_E^2$

- $\beta = A - B$ is positive iff

$$\tau < T \ell\, \delta x\, \rho_E^2 \tag{P_β}$$

- The sign of $\xi = -B$ is the one of $-\tau$

Case $\tau < 0$

If $\tau < 0$ and (P_α) holds, $f(x) = \alpha(1-\alpha)x + 2\beta\xi x^2$ is convex and is maximal on $[-1, 1]$ for $x = -1$ or $x = 1$

Therefore the model is stable if $f(-1) < f(1)$; this is simply

$$-\alpha(1-\alpha) < \alpha(1-\alpha)$$

that is always true since $\alpha > 0$ if (P_α) holds and $1 - \alpha > 0$ on $\tau < 0$

Therefore the system is stable for all $\tau < 0$

Case $\tau > 0$

Several cases have to be distinguished:

- $0 < \tau < \frac{1}{2} T \ell \, \delta x \, \rho_E^2 \qquad \alpha, 1-\alpha, \beta > 0, \ \ \xi < 0$

  $f(x) = \alpha(1-\alpha)x + 2\beta\xi x^2$ is concave and maximal for $x_0 = -\frac{\alpha(1-\alpha)}{4\beta\xi} > 0$;
  The model is stable if $x_0 > 1$, this is $\delta t < \frac{\delta x \, T}{\ell} \left(1 - \frac{2\tau}{T \ell \, \delta x \, \rho_E^2}\right)$

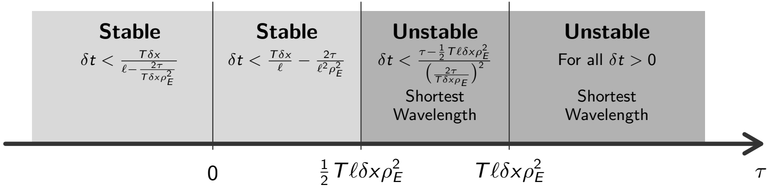
- $\frac{1}{2} T \ell \, \delta x \, \rho_E^2 < \tau < T \ell \, \delta x \, \rho_E^2 \qquad \alpha, \beta > 0, \ \ 1-\alpha, \xi < 0$

  We have $f(-1) > f(1)$ therefore the model is unstable; f maximal for $x_0 < -1$ (shortest wave) if $\delta t < \frac{\delta x \, T}{2\ell} \left(\frac{2\tau}{T \ell \, \delta x \, \rho_E^2} - 1\right) \left(\frac{2\tau}{T \ell \, \delta x \, \rho_E^2}\right)^{-2}$

- $\tau > T \ell \, \delta x \, \rho_E^2 \qquad \alpha > 0, \ \ 1-\alpha, \beta, \xi < 0$

  Unstable $\forall \, \delta t$ with shortest wavelength since f convex and $f(-1) > f(1)$

Scheme (D1) — Summary



$\rightarrow$ The same conditions as the continuous macroscopic model for:

$$\delta x \to 0 \qquad (\text{and } \delta t \to 0 \text{ such that } \delta t / \delta x \to 0)$$

Stability analysis for the schemes (D2) and (D3)

Affine speed function $V(\rho) = \frac{1}{T}(1/\rho - \ell)$, with $T > 0$ the time gap between the vehicles and $\ell > 0$ their size — Godunov scheme is $G(x, y) = \frac{1}{T}(1 - y\ell)$

The schemes (D2) and (D3) are

$$F_2(\rho_i, \rho_{i+1}, \rho_{i+2}, \rho_{i-1}) = \rho_i + \frac{\delta t \ell}{\delta x T}\left(\frac{\rho_{i+1}}{1 - \frac{\tau}{\delta x T}\left(\frac{1}{\rho_{i+2}} - \frac{1}{\rho_{i+2}}\right)} - \frac{\rho_i}{1 - \frac{\tau}{\delta x T}\left(\frac{1}{\rho_{i+1}} - \frac{1}{\rho_i}\right)}\right)$$

$$F_3(\rho_i, \rho_{i+1}, \rho_{i+2}, \rho_{i-1}) = \rho_i + \frac{\delta t \ell}{\delta x T}\left(\rho_{i+1} - \rho_i + \frac{\tau}{\delta x T}\left(\frac{\rho_{i+1} - \rho_{i+2}}{\rho_i} - \frac{\rho_i - \rho_{i+1}}{\rho_{i-1}}\right)\right)$$

By construction, both gives (with $A = \frac{\delta t\, \ell}{\delta x\, T}$ and $B = \frac{\tau}{T\, \delta x\, \rho_E}$)

$$\begin{aligned} \alpha &= 1 - A(1 + B) & \beta &= A(1 + 2B) \\ \gamma &= -AB & \xi &= 0 \end{aligned}$$

Signs of the partial derivative $\alpha\ \beta\ \gamma$

$A = \frac{\delta t\, \ell}{\delta x\, T}$ and $B = \frac{\tau}{\delta x\, T \rho_E}$

- $\alpha = 1 - A(1 + B)$ is positive if

$$\delta t < \frac{\delta x\, T}{\ell} \left(1 + \frac{\tau}{T\, \delta x\, \rho_E}\right)^{-1} \tag{P_α}$$

  if $\tau > -T\, \delta x\, \rho_E$, or for all $\delta t \geq 0$ if $\tau \leq -T\, \delta x\, \rho_E$

- $\beta = A(1 + 2B)$ is positive iff

$$\tau > -\frac{1}{2} T\, \delta x\, \rho_E \tag{P_β}$$

  Moreover $1 - \beta > 0$ if (P_α) holds
- The sign of $\gamma = -AB$ is the one of $-\tau$

Case $\tau < 0$

If $\tau < 0$ and (P_α) holds, $f(x) = \beta(1-\beta)x + 2\alpha\gamma x^2$ is convex is maximal on $[-1, 1]$ for $x = -1$ or $x = 1$

Therefore the model is stable if $f(-1) < f(1)$; this is

$$\tau > -\frac{1}{2} T\,\delta\mathrm{x}\,\rho_E \qquad \text{and} \qquad \delta\mathrm{t} < \frac{\delta\mathrm{x}\,T}{\ell}\left(1 + \frac{2\tau}{T\,\delta\mathrm{x}\,\rho_E}\right)^{-1}$$

The condition on $\delta\mathrm{t}$ is weaker than (P_α)

If $\tau \leq -\frac{1}{2} T\,\delta\mathrm{x}\,\rho_E$ then the system is unstable at the shortest wave-length frequency $\cos^{-1}(-1)$

A sufficiently condition for that the finite system produces the frequency $\cos^{-1}(-1)$ is simply $N \geq 2$

Case $\tau > 0$

(P_α) holds then $f(x) = \beta(1-\beta)x + 2\alpha\gamma x^2$ is concave and is maximum at $\arg\sup_x f(x) = x_0 = -\frac{\beta(1-\beta)}{4\alpha\gamma} > 0$

We know that $\lambda_0^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta + f(1) = 1$ (case $l = 0$)

Therefore the model is stable if $x_0 > 1$; this is

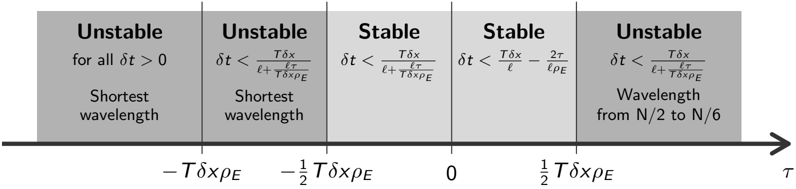
$$\tau < \frac{1}{2} T\,\delta\mathrm{x}\,\rho_E \qquad \text{and} \qquad \delta\mathrm{t} < \frac{\delta\mathrm{x}\,T}{\ell}\left(1 - \frac{2\tau}{T\,\delta\mathrm{x}\,\rho_E}\right)$$

The condition on $\delta\mathrm{t}$ is stronger than (P_α)

If $\tau \geq \frac{1}{2} T\,\delta\mathrm{x}\,\rho_E$ then the system is unstable at the frequency $\cos^{-1}(x_0)$ that is reachable in the finite system if $N > 2\pi/\cos^{-1}(x_0)$

We have $x_0 \underset{\delta\mathrm{t}\to 0}{\longrightarrow} \frac{1}{2} + \frac{T\,\delta\mathrm{x}\,\rho_E}{4\tau}$ going from 1 to 1/2 according to τ (long-waves)

Schemes (D2) and (D3) — Summary



$\rightarrow$ The same conditions as the microscopic model for:

$$\delta t \to 0 \qquad \text{and} \qquad \delta x = 1/\rho_E = \text{mean spacing}$$

Outline

Microscopic model

Micro–Macro derivation

Numerical schemes

Stability analysis

Simulation results

Simulation

Models

Microscopic: Euler explicit scheme

Macroscopic: Godunov/Godonov scheme (D3)

Setting

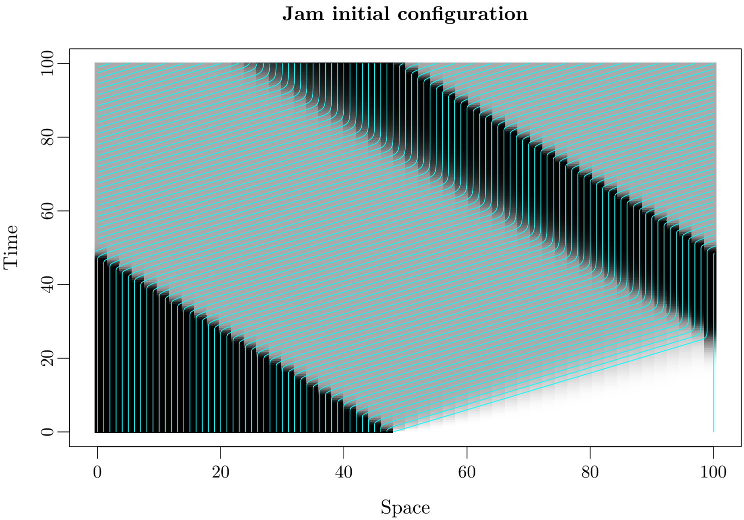
$\rho_E = 2$, $\delta x = 1/\rho_E$, $\delta t = 1\text{e-}2$, $V : \rho \mapsto \max\{\min\{2, 1/\rho - 1\}\}$

Environment

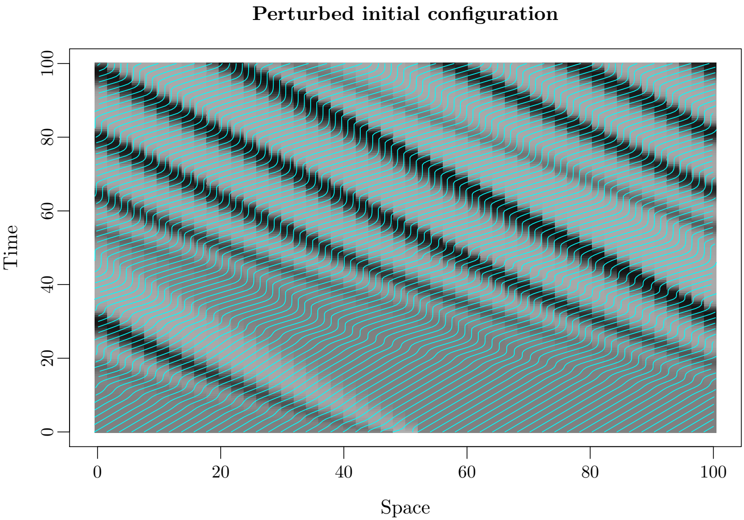
Ring (periodic conditions)

Initial condition: jam, random, perturbed

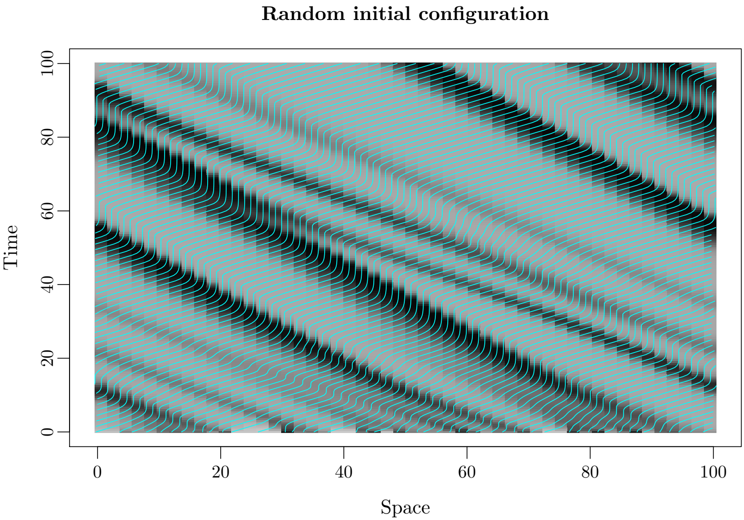
Trajectories



Trajectories



Trajectories



Fundamental diagram